

On antipodal spherical t -designs of degree s with $t \geq 2s - 3$

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Abstract

We prove that if X is a spherical t -design and s -distance set with $t \geq 2s - 3$, then X has the structure of Q-polynomial association scheme of class s . Also, we describe the parameters of the association scheme.

1 Introduction

Delsarte-Goethals-Seidel [9] studied finite subsets X in the unit sphere $S^{n-1} = \{x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n \mid x_1^2 + x_2^2 + \dots + x_n^2 = 1\}$ from the viewpoint of algebraic combinatorics. Here the following two parameters $s = s(X)$ and $t = t(X)$ play important roles. The s is called the degree of X and is defined as the number of distinct distances between two distinct elements of X , (then X is called an s -distance set), while t is called the strength of X and is usually defined as the largest t such that X becomes a t -design. Note that when we say X is a t -design, the strength of X might be larger than $s(X)$. So our use of t is slightly ambiguous, but this convention is very useful and we believe that serious confusion will not occur. The important results due to Delsarte-Goethals-Seidel [9] are as follows (the leader is referred to [9, 1] for the definition of undefined terminologies):

- (i) We always have $t \leq 2s$.
- (ii) If $t \geq s - 1$, then X is distance invariant.
- (iii) If $t \geq 2s - 2$, then X has the structure of Q-polynomial association scheme of class s .
- (iv) $t = 2s$ if and only if X is a tight $2s$ -design.
- (v) $t = 2s - 1$ and X is antipodal, i.e., if $x \in X$ then $-x \in X$, if and only if X is a tight $(2s - 1)$ -design.

Note that tight t -designs are classified except for $t \neq 4, 5, 7$ (see [3, 4, 7]) and there are some recent development on the study of tight t -designs with $t = 4, 5, 7$ (cf. [6].) So, it is important to study, or to classify, (or to show the nonexistence for the case of large t and $n \geq 3$), X with t close to $2s$.

In this paper, we study the case $t = 2s - 3$ with the additional assumption that X is antipodal, and show that similar properties for the general non-antipodal case with $t \geq 2s - 2$ hold true (cf. (iii) above). Namely, in this paper, we prove the following theorems.

Theorem 1.1 *Let X be a spherical t -design and s -distance set. Assume X is antipodal and $t \geq 2s - 3$. Then X has the structure of an association scheme of class s .*

Theorem 1.2 *Let $P = (P_j(i))$, $Q = (Q_j(i))$ be the first and second eigen matrices and $q_{i,j}^k$, $0 \leq i, j, k \leq s$, be the Krein numbers of the association scheme given in Theorem 1.1. Then by a suitable ordering of the adjacency matrices D_0, D_1, \dots, D_s and the basis of primitive idempotents E_0, E_1, \dots, E_s we obtain the followings.*

- (1) $P_1(i) = (-1)^i$ for any i with $0 \leq i \leq s$.
- (2) $Q_j(2i+1) = (-1)^j Q_j(2i)$ for any i and j with $1 \leq 2i+1 \leq s$ and $0 \leq j \leq s$. If s is even, then $Q_j(s) = 0$ for any odd integer j satisfying $1 \leq j \leq s-1$.
- (3) $m_i = Q_i(0) = \binom{n+i-1}{i} - \binom{n+i-3}{s-2}$ for $0 \leq i \leq s-2$,
 $m_{s-1} = Q_{s-1}(0) = \frac{|X|}{2} - \binom{n+s-4}{s-3}$ and
 $m_s = \frac{|X|}{2} - \binom{n+s-3}{s-2}$.
- (4) $q_{\mu,i}^j = 0$ for any μ, i, j satisfying $0 \leq \mu, i, j \leq s$ and $\mu + i + j = 0$.
- (5) The dual intersection matrix $B_1^* = (q_{1,i}^j)$ is tri-diagonal, that is the association scheme is a Q -polynomial scheme. More precisely

$$B_1^* = \begin{bmatrix} * & c_1^* & \cdots & c_{s-1}^* & n \\ 0 & 0 & \cdots & \cdots & 0 \\ n & b_1^* & \cdots & b_{s-1}^* & * \end{bmatrix}$$

where

$$c_i^* = q_{1,i-1}^i = \frac{ni}{n+2i-2}, \text{ for } i = 1, \dots, s-2,$$

$$b_i^* = q_{1,i}^{i-1} = \frac{n(n+i-3)}{n+2i-4} \text{ for } i = 1, \dots, s-3,$$

$$c_{s-1}^* = q_{1,s-2}^{s-1} = \frac{2n(n-1)(n+s-4)!}{(s-2)!(n-1)!|X|-2(s-2)(n+s-4)!}$$

$$b_{s-2}^* = q_{1,s-1}^{s-2} = \frac{n(n+s-4)}{n+2s-6},$$

and

$$b_{s-1}^* = q_{1,s}^{s-1} = \frac{(s-2)!n!|X|-2n(n+s-3)!}{(s-2)!(n-1)!|X|-2(s-2)(n+s-4)!}.$$

In §2, we give some basic facts on spherical t -designs and s -distance sets. In §3, we prove that X has the structure of an association scheme of class s . In §4 we prove Theorem 1.2.

2 Some basic facts about spherical t -designs

We use the method given in the paper by Delsarte-Goethals-Seidel [9] to prove our main results. First we introduce some definition and notations. We denote by $\tilde{Q}_l(x)$ the Gegenbauer polynomial of degree l attached to the unit sphere $S^{n-1} \subset \mathbb{R}^n$. We use the notation \tilde{Q} in order to distinguish it from the second eigenmatrix Q . Here we use the normalization so that $\tilde{Q}_l(1) = \dim(\text{Harm}_l(\mathbb{R}^n))$ holds. It is well known that the Gegenbauer polynomials satisfy the following equations (see [9, 1]).

$$\frac{l+1}{n+2l}\tilde{Q}_{l+1}(x) = x\tilde{Q}_l(x) - \frac{n+l-3}{n+2l-4}\tilde{Q}_{l-1}(x) \quad \text{for any integer } l \geq 0 \text{ and} \quad (2.1)$$

$$\tilde{Q}_i(x)\tilde{Q}_j(x) = \sum_{\substack{k=|i-j| \\ k \equiv i+j \pmod{2}}}^{i+j} q_k(i, j)\tilde{Q}_k(x) \quad \text{for any integer } i, j, k \geq 0, \quad (2.2)$$

where $q_k(i, j)$ is a nonnegative real number. Let

$$x^\lambda = \sum_{l=0}^{\lambda} f_{\lambda, l} \tilde{Q}_l(x) \quad (2.3)$$

be the Gegenbauer expansion of x^λ for any non negative integer λ . For each pair λ, μ of nonnegative integers, we define a polynomial by $F_{\lambda, \mu}(x) = \sum_{l=0}^{\min\{\lambda, \mu\}} f_{\lambda, l} f_{\mu, l} \tilde{Q}_l(x)$. We denote by $\mathbf{x} \cdot \mathbf{y}$ the canonical inner product between the vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. For a finite subset $X \subset S^{n-1}$ we define $A(X) = \{\mathbf{x} \cdot \mathbf{y} \mid \mathbf{x}, \mathbf{y} \in X, \mathbf{x} \neq \mathbf{y}\}$ and $A'(X) = A(X) \cup \{1\}$. Let $s = |A(X)|$. Then X is called an s -distance set. Now we express $A(X) = \{\alpha_1, \alpha_2, \dots, \alpha_s\}$. Let $\alpha_0 = 1$. We define matrices D_i indexed by X by

$$D_i(\mathbf{x}, \mathbf{y}) = \begin{cases} 1 & \text{if } \mathbf{x} \cdot \mathbf{y} = \alpha_i, \\ 0 & \text{if } \mathbf{x} \cdot \mathbf{y} \neq \alpha_i. \end{cases} \quad (2.4)$$

Let $\text{Harm}(\mathbb{R}^n)$ be the vector space of the harmonic polynomials on n variables $\mathbf{x} = (x_1, x_2, \dots, x_n)$. We define a positive definite innerproduct on $\text{Harm}(\mathbb{R}^n)$ by

$$\langle \varphi, \psi \rangle = \frac{1}{|S^{n-1}|} \int_{S^{n-1}} \varphi(\mathbf{x}) \psi(\mathbf{x}) d\sigma(\mathbf{x})$$

for $\varphi, \psi \in \text{Harm}(\mathbb{R}^n)$. Let $\text{Harm}_l(\mathbb{R}^n)$ be the subspace of $\text{Harm}(\mathbb{R}^n)$ generated by homogeneous harmonic polynomial of degree l . Let $\varphi_{l,1}, \dots, \varphi_{l,h_l}$ be an orthonormal basis of $\text{Harm}_l(\mathbb{R}^n)$ with respect to the inner product given above. The following addition formula is well known.

Addition formula

$$\sum_{i=1}^{h_l} \varphi_{l,i}(\mathbf{x}) \varphi_{l,i}(\mathbf{y}) = \tilde{Q}_l(\mathbf{x} \cdot \mathbf{y}) \quad (2.5)$$

holds for any $\mathbf{x}, \mathbf{y} \in S^{n-1}$.

Let H_l be the matrix indexed by X and $\{\varphi_{l,i} \mid 1 \leq i \leq h_l\}$, whose $(x, \varphi_{l,i})$ -entry is $\varphi_{l,i}(\mathbf{x})$. The following proposition is well known.

Proposition 2.1 *Let X be a spherical t -design and s -distance set. The notation is given as before.*

- (1) ${}^tH_k H_l = \delta_{k,l} I$ for any nonnegative integers k, l satisfying $0 \leq k + l \leq t$.
- (2) $H_k {}^tH_k = \sum_{i=0}^s \tilde{Q}_k(\alpha_i) D_i$ for any nonnegative integer k .
- (3) $H_k {}^tH_k \neq 0$ for $k = 0, 1, \dots, s$.

Proof Let us compute the $(\varphi_{k,i}, \varphi_{l,j})$ -entry of ${}^tH_k H_l$. Since X is a spherical t -design, we obtain

$$\sum_{\mathbf{x} \in X} \varphi_{k,i}(\mathbf{x}) \varphi_{l,j}(\mathbf{x}) = \frac{|X|}{|S^{n-1}|} \int_{S^{n-1}} \varphi_{k,i}(\mathbf{x}) \varphi_{l,j}(\mathbf{x}) d\sigma(\mathbf{x}) = |X| \delta_{k,l} \delta_{i,j}.$$

This implies (1). Let us compute the (\mathbf{x}, \mathbf{y}) -entry of $H_k {}^tH_k$. Then addition formula (2.5) implies

$$\sum_{i=1}^{h_k} \varphi_{i,k}(\mathbf{x}) \varphi_{i,k}(\mathbf{y}) = \tilde{Q}_k(\mathbf{x} \cdot \mathbf{y}).$$

This implies (2). Since $\tilde{Q}_k(x)$ is a polynomial of degree $k \leq s$ and $\alpha_0, \alpha_1, \dots, \alpha_s$ are distinct $s+1$ real numbers, we must have an α_i satisfying $\tilde{Q}_k(\alpha_i) \neq 0$. This implies (3). ■

3 Proof of Theorem 1.1

Let X be a spherical t -design and an s -distance set. assume X is antipodal. Then, if s is odd, then by arranging the numbering of the elements in $A(X) = \{\alpha_i \mid 1 \leq i \leq s\}$ we may assume $\alpha_1 = -1$, $\alpha_{2i+1} = -\alpha_{2i} \neq 0$ for $i = 0, \dots, \frac{s-1}{2}$. If s is even, we may assume $\alpha_1 = -1$, $\alpha_{2i+1} = -\alpha_{2i} \neq 0$ for $i = 0, \dots, \frac{s}{2} - 1$ and $\alpha_s = 0$. Note that $\alpha_0 = 1$. For $\mathbf{x}, \mathbf{y} \in X$ and $\alpha, \beta \in A'(X)$, we define

$$p_{\alpha,\beta}(\mathbf{x}, \mathbf{y}) = |\{z \mid \mathbf{x} \cdot z = \alpha \text{ and } z \cdot \mathbf{y} = \beta\}|.$$

Lemma 3.1 *Definitions and notation are given as before. Let $\mathbf{x} \cdot \mathbf{y} = \gamma$. Then $p_{\alpha,\beta}(\mathbf{x}, \mathbf{y})$ depends only on α, β and γ and does not depend on the choice of \mathbf{x}, \mathbf{y} satisfying $\mathbf{x} \cdot \mathbf{y} = \gamma$.*

Proof We have the following

$$p_{\alpha_i, \alpha_0}(\mathbf{x}, \mathbf{y}) = p_{\alpha_i, \alpha_0}(\mathbf{x}, \mathbf{y}) = \begin{cases} 1 & \text{if } \alpha_i = \gamma, \\ 0 & \text{otherwise.} \end{cases} \quad (3.1)$$

$$p_{\alpha_i, \alpha_1}(\mathbf{x}, \mathbf{y}) = p_{\alpha_i, \alpha_1}(\mathbf{x}, \mathbf{y}) = \begin{cases} 1 & \text{if } \alpha_i = -\gamma, \\ 0 & \text{otherwise.} \end{cases} \quad (3.2)$$

Let $\mathbf{x}, \mathbf{y} \in X$ and $\mathbf{x} \cdot \mathbf{y} = \gamma$. Then for any λ and μ satisfying $0 \leq \lambda, \mu \leq s-2$ we compute the (\mathbf{x}, \mathbf{y}) -entry of the matrix

$$\left(\sum_{k=0}^{\lambda} f_{\lambda,k} H_k {}^tH_k \right) \left(\sum_{l=0}^{\mu} f_{\mu,l} H_l {}^tH_l \right)$$

in two different ways, where $f_{\lambda,k}(0 \leq k \leq \lambda)$, $f_{\mu,l}(0 \leq l \leq \mu)$ are defined in (2.3). First use Proposition 2.1 (1), and then use Proposition 2.1 (2), then we obtain

$$\begin{aligned} & ((\sum_{k=0}^{\lambda} f_{\lambda,k} H_k {}^t H_k) (\sum_{l=0}^{\mu} f_{\mu,l} H_l {}^t H_l))(\mathbf{x}, \mathbf{y}) = |X| \sum_{k=0}^{\min\{\lambda, \mu\}} f_{\lambda,k} f_{\mu,k} (H_k {}^t H_k)(\mathbf{x}, \mathbf{y}) \\ & = |X| \sum_{k=0}^{\min\{\lambda, \mu\}} f_{\lambda,k} f_{\mu,k} \tilde{Q}_k(\gamma) = |X| F_{\lambda, \mu}(\gamma). \end{aligned} \quad (3.3)$$

Next, first apply Proposition 2.1 (2) and then use (2.3). Then we obtain

$$\begin{aligned} & ((\sum_{k=0}^{\lambda} f_{\lambda,k} H_k {}^t H_k) (\sum_{l=0}^{\mu} f_{\mu,l} H_l {}^t H_l))(\mathbf{x}, \mathbf{y}) \\ & = ((\sum_{k=0}^{\lambda} f_{\lambda,k} \sum_{i=0}^s Q_k(\alpha_i) D_i) (\sum_{l=0}^{\mu} f_{\mu,l} \sum_{j=0}^s Q_l(\alpha_j) D_j))(\mathbf{x}, \mathbf{y}) \\ & = \sum_{k=0}^{\lambda} \sum_{l=0}^{\mu} \sum_{i=0}^s \sum_{j=0}^s f_{\lambda,k} f_{\mu,l} Q_k(\alpha_i) Q_l(\alpha_j) \sum_{\mathbf{z} \in X} D_i(\mathbf{x}, \mathbf{z}) D_j(\mathbf{z}, \mathbf{y}) \\ & = \sum_{i=0}^s \sum_{j=0}^s \sum_{k=0}^{\lambda} \sum_{l=0}^{\mu} f_{\lambda,k} Q_k(\alpha_i) f_{\mu,l} Q_l(\alpha_j) p_{\alpha_i, \alpha_j}(\mathbf{x}, \mathbf{y}) \\ & = \sum_{i=0}^s \sum_{j=0}^s \alpha_i^{\lambda} \alpha_j^{\mu} p_{\alpha_i, \alpha_j}(\mathbf{x}, \mathbf{y}) = \sum_{i=2}^s \sum_{j=2}^s \alpha_i^{\lambda} \alpha_j^{\mu} p_{\alpha_i, \alpha_j}(\mathbf{x}, \mathbf{y}) + \sum_{i=0}^1 \sum_{j=2}^s \alpha_i^{\lambda} \alpha_j^{\mu} p_{\alpha_i, \alpha_j}(\mathbf{x}, \mathbf{y}) \\ & + \sum_{i=2}^s \sum_{j=0}^1 \alpha_i^{\lambda} \alpha_j^{\mu} p_{\alpha_i, \alpha_j}(\mathbf{x}, \mathbf{y}) + \sum_{i=0}^1 \sum_{j=0}^1 \alpha_i^{\lambda} \alpha_j^{\mu} p_{\alpha_i, \alpha_j}(\mathbf{x}, \mathbf{y}) \\ & = \sum_{i=2}^s \sum_{j=2}^s \alpha_i^{\lambda} \alpha_j^{\mu} p_{\alpha_i, \alpha_j}(\mathbf{x}, \mathbf{y}) \\ & + \begin{cases} \gamma^{\mu} + (-1)^{\lambda} (-\gamma)^{\mu} + \gamma^{\lambda} + (-\gamma)^{\lambda} (-1)^{\mu} & \text{if } \gamma \neq \alpha_0, \alpha_1 \\ \delta_{\gamma, \alpha_0} + (-1)^{\mu} \delta_{\gamma, \alpha_1} + (-1)^{\lambda} \delta_{\gamma, \alpha_1} + (-1)^{\lambda+\mu} \delta_{\gamma, \alpha_0} & \text{if } \gamma \in \{\alpha_0, \alpha_1\} \end{cases} \end{aligned} \quad (3.4)$$

Thus (3.4) and (3.4) yield a system of linear equation whose indeterminates are $p_{\alpha_i, \alpha_j}(\mathbf{x}, \mathbf{y})$, $2 \leq i, j \leq s$ and the coefficient matrix is $W \otimes W$, where

$$W = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ \alpha_2 & \alpha_3 & \cdots & \alpha_s \\ \alpha_2^2 & \alpha_3^2 & \cdots & \alpha_s^2 \\ \vdots & \vdots & \vdots & \vdots \\ \alpha_2^{s-2} & \alpha_3^{s-2} & \cdots & \alpha_s^{s-2} \end{bmatrix}$$

Since W is invertible, $p_{\alpha_i, \alpha_j}(\mathbf{x}, \mathbf{y})$, ($2 \leq i, j \leq s$) are uniquely determined by α_i, α_j and γ and does not depend on the choice of \mathbf{x}, \mathbf{y} satisfying $\mathbf{x} \cdot \mathbf{y} = \gamma$. This completes the proof. ■

Lemma 3.1 implies that the spherical t -design given in Theorem 1.1 has the structure of an association scheme.

4 Proof of Theorem 1.2

First we will show the following proposition.

Proposition 4.1 *The definition and notation are given as before. Let $F_i = \frac{1}{|X|}H_i {}^tH_i$ for $i = 0, \dots, s-2$. Then $\{F_0, F_1, \dots, F_{s-2}\}$ is a subset of the basis of the primitive idempotents of the Bose-Mesner algebra $\mathfrak{A} = \langle D_0, D_1, \dots, D_s \rangle$.*

Proof By definition $F_0 = \frac{1}{|X|}J = E_0$, where J is the matrix whose elements are all 1. Proposition 2.1 implies that $F_i \in \mathfrak{A}$ and $F_i F_j = \delta_{i,j} F_i$ for any $0 \leq i, j \leq s-2$. Let $F_{s-1} = \frac{1}{|X|}H_{s-1} {}^tH_{s-1}$ and $F_s = I - \sum_{i=0}^{s-2} F_i$. Then $F_{s-1}, F_s \in \mathfrak{A}$. Moreover $F_{s-1} F_i = F_s F_i = 0$ and $F_s^2 = F_s$ holds for any $i = 0, \dots, s-2$. Since F_0, F_1, \dots, F_{s-2} are idempotents, each F_i is a partial some of the primitive idempotents E_0, E_1, \dots, E_s of \mathfrak{A} . Assume that there exists an $F_i \notin \{E_0, E_1, \dots, E_s\}$, then we have decomposition $F_i = E + E'$, satisfying $E, E' \neq 0$, $E^2 = E$, $(E')^2 = E'$ and $EE' = 0$. Then, $F_i E = E$ and $F_i E' = E'$ implies $F_{s-1} E = F_{s-1} E'$ and $F_s E = F_s E' = 0$. On the other hand $F_0, F_1, \dots, F_{i-1}, E, E', F_{i+1}, \dots, F_{s-2}$ are linearly independent. Since Proposition 2.1 (3) implies $F_{s-1} \neq 0$, $\dim(\langle F_{s-1}, F_s \rangle) = 1$. Hence there exists a real number c satisfying $F_s = cF_{s-1}$. This implies

$$D_0 - \frac{1}{|X|} \sum_{i=0}^{s-2} \sum_{j=0}^s \tilde{Q}_i(\alpha_j) D_j = c \frac{1}{|X|} \sum_{j=0}^s \tilde{Q}_{s-1}(\alpha_j) D_j.$$

Then we obtain

$$c\tilde{Q}_{s-1}(\alpha_j) + \sum_{i=0}^{s-2} \tilde{Q}_i(\alpha_j) = 0$$

for any $j = 1, \dots, s$. However $c\tilde{Q}_{s-1}(x) + \sum_{i=0}^{s-2} \tilde{Q}_i(x)$ is a polynomial of degree at most $s-1$ and $\alpha_1, \dots, \alpha_s$ are distinct to each other, this is a contradiction. Hence $F_i \in \{E_j \mid 0 \leq j \leq s\}$ for any $0 \leq i \leq s-2$. ■

Proof of Theorem 1.2 (1)

Let E_0, E_1, \dots, E_d be the basis of primitive idempotents of \mathfrak{A} satisfying $E_i = \frac{1}{|X|}H_i {}^tH_i$ for $i = 0, 1, \dots, s-2$. We note that we defined $\alpha_0 = 1$, $\alpha_1 = -1$, $\alpha_{2i+1} = -\alpha_{2i}$ for $0 \leq 2i+1 \leq s$, and if s is even $\alpha_s = 0$. Let B_1 be the intersection matrix of \mathfrak{A} whose (i, j) -entry is defined by $B_1(i, j) = p_{1,i}^j$. Then we have

$$B_1 = \begin{bmatrix} S & 0 & \cdots & 0 \\ 0 & S & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & S \end{bmatrix} \quad \text{if } s \text{ is odd,} \quad B_1 = \begin{bmatrix} S & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & S & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix} \quad \text{if } s \text{ is even.} \quad (4.1)$$

where $S = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Hence the matrix B_1 has the eigenvalue 1 with multiplicity $\lceil \frac{s+2}{2} \rceil$ and -1 with multiplicity $\lceil \frac{s+1}{2} \rceil$. Let us compute $D_1 D_j$ in two different ways. First we have

$$D_1 D_j = \sum_{l=0}^s p_{1,j}^l D_l = \sum_{l=0}^s p_{1,j}^l \sum_{\mu=0}^s P_l(\mu) E_\mu = \sum_{\mu=0}^s \left(\sum_{l=0}^s p_{1,j}^l P_l(\mu) \right) E_\mu.$$

On the other hand

$$D_1 D_j = \left(\sum_{\mu=0}^s P_1(\mu) E_\mu \right) \left(\sum_{\lambda=0}^s P_j(\lambda) E_\lambda \right) = \sum_{\mu=0}^s P_1(\mu) P_j(\mu) E_\mu$$

Hence we obtain

$$\sum_{l=0}^s p_{1,j}^l P_l(\mu) = P_j(\mu) P_1(\mu) \quad (4.2)$$

for any $j, \mu = 0, 1, \dots, s$. Let M_1 be a diagonal matrix whose diagonal entries are defined by $M_1(i, i) = P_1(i)$ for $i = 0, 1, \dots, s$. Then (4.2) implies

$$B_1 {}^t P = {}^t P M_1. \quad (4.3)$$

Hence ${}^t P M_1 ({}^t P)^{-1} = B_1$ holds. This implies that $\{P_1(0), P_1(1), \dots, P_1(s)\}$ is the set of eigenvalues of B_1 . Hence $P_1(i) = 1$ or -1 for any $0 \leq i \leq s$ and $|\{i \mid P_1(i) = 1\}| = \lceil \frac{s+2}{2} \rceil$ and $|\{i \mid P_1(i) = -1\}| = \lceil \frac{s+1}{2} \rceil$ holds. On the other hand, $E_i = \frac{1}{|X|} H_i {}^t H_i = \frac{1}{|X|} \sum_{j=0}^s \tilde{Q}_i(\alpha_j) D_j$ for $0 \leq i \leq s-2$, and (4.1) imply

$$\begin{aligned} D_1 E_i &= \frac{1}{|X|} \sum_{j=0}^s \tilde{Q}_i(\alpha_j) D_1 D_j = \frac{1}{|X|} \sum_{l=0}^s \left(\sum_{j=0}^s \tilde{Q}_i(j) p_{1,j}^l \right) D_l \\ &= \frac{1}{|X|} \sum_{l=0}^s \left(\sum_{j=0}^s \tilde{Q}_i(j) B_1(l, j) \right) D_l = \frac{1}{|X|} \sum_{l=0}^s (B_1 \tilde{Q})(l, i) D_l \\ &= \frac{1}{|X|} \left(\tilde{Q}_i(\alpha_1) D_0 + \tilde{Q}_i(\alpha_0) D_1 + \dots + \tilde{Q}_i(\alpha_{2j+1}) D_{2j} + \tilde{Q}_i(\alpha_{2j}) D_{2j+1} + \dots \right) \\ &= \frac{1}{|X|} \sum_{j=0}^s \tilde{Q}_i(-\alpha_j) D_j \\ &= (-1)^i E_i, \end{aligned} \quad (4.4)$$

$$D_1 E_i = \sum_{l=0}^s P_1(l) E_l E_i = P_1(i) E_i. \quad (4.5)$$

Hence we obtain $P_1(i) = (-1)^i$ for $0 \leq i \leq s-2$. Hence by arranging the ordering of E_{s-1} and E_s , we obtain $P_1(i) = (-1)^i$ for any $0 \leq i \leq s$. This completes the proof.

Proof of Theorem 1.2 (2)

If $0 \leq j \leq s-2$, then $Q_j(l) = \tilde{Q}_j(\alpha_l)$. By definition $\alpha_{2i+1} = -\alpha_{2i}$ for $0 \leq i \leq \frac{s-1}{2}$. Hence

$Q_j(2i+1) = \tilde{Q}_j(\alpha_{2i+1}) = \tilde{Q}_j(-\alpha_{2i}) = (-1)^j \tilde{Q}_j(\alpha_{2i}) = (-1)^j Q_j(2i)$. If s is even, then $\alpha_s = 0$. Hence $Q_j(s) = \tilde{Q}_j(\alpha_s) = \tilde{Q}_j(0) = 0$ for any odd integer j satisfying $0 \leq j \leq s-2$. Let us consider the condition $QP = |X|I$. If $1 \leq 2i+1 \leq s$, we have the following equations.

$$\sum_{j=0}^s Q_j(2i) = \delta_{i,0}|X|, \quad (4.6)$$

$$\sum_{j=0}^s Q_j(2i)P_1(j) = \sum_{j=0}^s (-1)^j Q_j(2i) = 0, \quad (4.7)$$

$$\sum_{j=0}^s Q_j(2i+1)P_0(j) = \sum_{j=0}^{s-2} (-1)^j Q_j(2i) + Q_{s-1}(2i+1) + Q_s(2i+1) = 0, \quad (4.8)$$

$$\begin{aligned} \sum_{j=0}^s Q_j(2i+1)P_1(j) &= \sum_{j=0}^s (-1)^j Q_j(2i+1) \\ &= \sum_{j=0}^{s-2} Q_j(2i) + (-1)^{s-1} Q_{s-1}(2i+1) + (-1)^s Q_s(2i+1) = \delta_{i,0}|X|, \end{aligned} \quad (4.9)$$

hold. Then (4.7) and (4.8) imply

$$(-1)^{s-1}(Q_{s-1}(2i) - Q_s(2i)) = Q_{s-1}(2i+1) + Q_s(2i+1). \quad (4.10)$$

Then (4.6) and (4.9) imply

$$Q_{s-1}(2i) + Q_s(2i) = (-1)^{s-1}(Q_{s-1}(2i+1) - Q_s(2i+1)). \quad (4.11)$$

(4.9) and (4.10) imply

$$Q_j(2i+1) = (-1)^j Q_j(2i) \quad \text{for } j = s-1 \text{ and } s.$$

Equations (4.6) and (4.7) imply

$$\sum_{j=0}^{\lfloor \frac{s}{2} \rfloor} Q_{2j}(2i) = \sum_{j=0}^{\lfloor \frac{s-1}{2} \rfloor} Q_{2j+1}(2i) = \delta_{i,0} \frac{|X|}{2}.$$

$$Q_{2\lfloor \frac{s}{2} \rfloor}(2i) = \delta_{i,0} \frac{|X|}{2} - \sum_{j=0}^{\lfloor \frac{s}{2} \rfloor - 1} Q_{2j}(2i), \quad (4.12)$$

$$Q_{2\lfloor \frac{s-1}{2} \rfloor + 1}(2i) = \delta_{i,0} \frac{|X|}{2} - \sum_{j=0}^{\lfloor \frac{s-1}{2} \rfloor - 1} Q_{2j+1}(2i) \quad (4.13)$$

Therefore, if s is even, then $\alpha_s = 0$. Hence $Q_{2j+1}(s) = \tilde{Q}_{2j+1}(\alpha_s) = \tilde{Q}_{2j+1}(0) = 0$ for any $j = 0, \dots, \frac{s}{2} - 2$. Hence (4.13) implies

$$Q_{s-1}(s) = - \sum_{j=0}^{\frac{s}{2}-2} Q_{2j+1}(s) = 0. \quad (4.14)$$

This completes the proof.

Proof of Theorem 1.2 (3)

If $i \leq s-2$, then $m_i = Q_i(0) = \tilde{Q}_i(1) = \binom{n+i-1}{i} - \binom{n+i-3}{i-2}$. If s is odd, then (4.12) and (4.13) imply

$$\begin{aligned}
m_{s-1} &= Q_{s-1}(0) = \frac{|X|}{2} - \sum_{j=0}^{\frac{s-3}{2}} m_{2j} \\
&= \frac{|X|}{2} - \sum_{j=0}^{\frac{s-3}{2}} \left(\binom{n+2j-1}{2j} - \binom{n+2j-3}{2j-2} \right) \\
&= \frac{|X|}{2} - \binom{n+s-4}{s-3}
\end{aligned} \tag{4.15}$$

$$\begin{aligned}
m_s &= Q_s(0) = \frac{|X|}{2} - \sum_{j=0}^{\frac{s-3}{2}} m_{2j+1} \\
&= \frac{|X|}{2} - \sum_{j=0}^{\frac{s-3}{2}} \left(\binom{n+2j}{2j+1} - \binom{n+2j-2}{2j-1} \right) \\
&= \frac{|X|}{2} - \binom{n+s-3}{s-2}
\end{aligned} \tag{4.16}$$

If s is even, then (4.12) and (4.13) imply

$$\begin{aligned}
m_{s-1} &= Q_{s-1}(0) = \frac{|X|}{2} - \sum_{j=0}^{\frac{s-4}{2}} m_{2j+1} \\
&= \frac{|X|}{2} - \sum_{j=0}^{\frac{s-4}{2}} \left(\binom{n+2j}{2j+1} - \binom{n+2j-2}{2j-1} \right) \\
&= \frac{|X|}{2} - \binom{n+s-4}{s-3}
\end{aligned} \tag{4.17}$$

$$\begin{aligned}
m_s &= Q_s(0) = \frac{|X|}{2} - \sum_{j=0}^{\frac{s-2}{2}} m_{2j} \\
&= \frac{|X|}{2} - \sum_{j=0}^{\frac{s-2}{2}} \left(\binom{n+2j-1}{2j} - \binom{n+2j-3}{2j-2} \right) \\
&= \frac{|X|}{2} - \binom{n+s-3}{s-2}
\end{aligned} \tag{4.18}$$

This completes the proof.

Proof of Theorem 1.2 (4)

We compute $(|X|E_i \circ |X|E_j) \circ D_l$ in two ways.

$$(|X|E_i \circ |X|E_j) \circ D_l = \left(\sum_{\mu=0}^s Q_i(\mu) D_\mu \right) \circ \left(\sum_{k=0}^s Q_j(k) D_k \right) \circ D_l = Q_i(l) Q_j(l) D_l \quad (4.19)$$

On the other hand we have

$$\begin{aligned} (|X|E_i \circ |X|E_j) \circ D_l &= \left(\sum_{\mu=0}^s q_{i,j}^\mu |X|E_\mu \right) \circ D_l = \left(\sum_{\mu=0}^s q_{i,j}^\mu \sum_{k=0}^s Q_\mu(k) D_k \right) \circ D_l \\ &= \sum_{\mu=0}^s q_{i,j}^\mu Q_\mu(l) D_l. \end{aligned} \quad (4.20)$$

(4.19) and (4.20) imply

$$Q_i(l) Q_j(l) = \sum_{\mu=0}^s q_{i,j}^\mu Q_\mu(l)$$

for any $0 \leq l \leq s$. In particular we have

$$Q_i(0) Q_j(0) = \sum_{\mu=0}^s q_{i,j}^\mu Q_\mu(0). \quad (4.21)$$

Since Theorem 1.2 (2) implies $Q_j(1) = (-1)^j Q_j(0)$, we obtain

$$Q_i(0) Q_j(0) = (-1)^{i+j} Q_i(1) Q_j(1) = (-1)^{i+j} \sum_{\mu=0}^s q_{i,j}^\mu Q_\mu(1) = \sum_{\mu=0}^s (-1)^{i+j+\mu} q_{i,j}^\mu Q_\mu(0). \quad (4.22)$$

Hence (4.21) and (4.22) imply

$$\sum_{\mu=0}^s ((-1)^{i+j+\mu} - 1) q_{i,j}^\mu Q_\mu(0) = 0.$$

Since $Q_\mu(0) = m_\mu > 0$ and $q_{i,j}^\mu \geq 0$ we must have $q_{i,j}^\mu = 0$ for any $0 \leq i, j, \mu \leq s$ satisfying $i + j + \mu$ is odd. This completes the proof.

Proof of Theorem 1.2 (5)

Now we are ready to prove that the association scheme given in Theorem 1.1 is Q-polynomial. For $0 \leq i \leq s-3$, we obtain

$$\begin{aligned} |X|E_1 \circ |X|E_i &= \sum_{l=0}^s \tilde{Q}_1(\alpha_l) D_l \circ \sum_{\mu=0}^s \tilde{Q}_i(\alpha_\mu) D_\mu = \sum_{l=0}^s \tilde{Q}_1(\alpha_l) \tilde{Q}_i(\alpha_l) D_l \\ &= \sum_{l=0}^s \sum_{k=i-1, i+1} q_k(1, i) \tilde{Q}_k(\alpha_l) D_l = \sum_{k=i-1, i+1} q_k(1, i) E_k. \end{aligned}$$

This implies $q_{1,i}^{i-1} = q_{i-1}(1, i) = \frac{n(n+i-3)}{n+2i-4}$ for any $1 \leq i \leq s-3$, $q_{1,i}^{i+1} = q_{i+1}(1, i) = \frac{n(i+1)}{n+2i}$ for any $0 \leq i \leq s-3$ and $q_{1,i}^j = 0$ for any i, j satisfying $0 \leq i \leq s-3$ and $0 \leq j \neq i-1, i+1$. Since Krein numbers satisfy $m_i q_{1,j}^i = m_j q_{1,i}^j$, and $m_i > 0$, we obtain $q_{1,s-1}^j = q_{1,s}^j = 0$ for $0 \leq j \leq s-3$ and $q_{1,s-2}^j = 0$ for $0 \leq j \leq s-4$. Also (1) implies $q_{1,s-2}^s = q_{1,s}^{s-2} = 0$. Thus B_1^* must be a tri-diagonal matrix and the association scheme is Q-polynomial.

Next, we determin the nonzero entries of B_1^* . Note that $F_{s-1} \in \langle E_{s-1}, E_s \rangle$.

$$\begin{aligned} |X|E_1 \circ |X|E_{s-2} &= \sum_{l=0}^s \tilde{Q}_1(\alpha_l)D_l \circ \sum_{\mu=0}^s \tilde{Q}_{s-2}(\alpha_\mu)D_\mu = \sum_{l=0}^s \tilde{Q}_1(\alpha_l)\tilde{Q}_{s-2}(\alpha_l)D_l \\ &= \sum_{l=0}^s \sum_{k=s-3, s-1} q_k(1, s-2)\tilde{Q}_k(\alpha_l)D_l = |X|q_{s-3}(1, s-2)E_{s-3} + |X|F_{s-1}. \end{aligned}$$

Hence $q_{1,s-2}^{s-3} = q_{s-3}(1, s-2) = \frac{n(n+s-5)}{n+2s-8}$. The formula $\sum_{i=0}^s q_{1,i}^k = m_1 (= n)$ for any $0 \leq k \leq s$ implies $q_{1,s-1}^s = n$ and $q_{1,s-1}^{s-2} = n - q_{1,s-3}^{s-2} = n - q_{s-2}(1, s-3) = \frac{n(n+s-4)}{n+2s-6}$. Then we obtain

$$q_{1,s-2}^{s-1} = \frac{m_{s-2}}{m_{s-1}} q_{1,s-1}^{s-2} = \frac{2n(n-1)(n+s-4)!}{(s-2)!(n-1)!|X| - 2(s-2)(n+s-4)!}.$$

Then

$$q_{1,s}^{s-1} = n - q_{1,s-2}^{s-1} = \frac{(s-2)!n!|X| - 2n(n+s-3)!}{(s-2)!(n-1)!|X| - 2(s-2)(n+s-4)!}.$$

This completes the proof. ■

5 Concluding Remarks

We consider the case of $t \geq 2s-3$ and $s=4$, the smallest nontrivial case, since the case $s=3$ is a kind of studied as the case of equiangular lines in Euclidean spaces, (cf. [10, 9].) The details of this case of $t \geq 5$ and $s=4$ was already mentioned in our previous paper [2], so we just summarize the results. Note that the connection of this case with tight Euclidean 7-designs is also mentioned in [2].

The first and the second eigen matrices P, Q and the dual intersection matrix B_1^* , as well as all the intersection matrices $B_i (i=0, 1, 2, 3, 4)$ of the corresponding association scheme are described in Appendix 1. (Note that they are all described by two parameters n and $N = |X|/2$.)

We have shown that if $n \geq 3$, then all the entries of P and Q are rational numbers. This gives a strong restriction on the possible pairs of n and N for which such association schemes may exist. It is also shown that for such an association scheme, if $A(X) = \{-1, -\alpha, 0, \alpha\}$ then $\alpha = \frac{1}{m}$ for a positive integer m . So, it is easy to see that if m is given, then there are only finitely many possible pairs of n and N . In [2] we have listed the possible pairs (n, N) for $m \leq 5$. We have succeeded in classifying those with

$m = 2, 3$ (see [2]) and found some examples with $m = 4$ and $N = 144$. We recognized that this association scheme comes from the real MUB(mutually unbiased bases) in \mathbb{R}^{16} , and subsequently we noticed that such examples are obtained for any integer $m = \frac{1}{2^{2r}}$. Namely, we have a family of Q-polynomial association schemes X with $|X| = 2^{4r} + 2^{2r+1}$ which gives the antipodal spherical design in S^{n-1} , $n = 2^{2r}$, of degree 4 and strength 5, with $A(X) = \{-1, -\frac{1}{2^{2r}}, 0, \frac{1}{2^{2r}}\}$. These Q-polynomial association schemes are "not" P -polynomial. (The explicit parameters are described in Appendix 2.) This family of association schemes was originally missing in the list of such association schemes in the home page of W. L. Martin (see [11]).

We also mention that explicit descriptions of possible parameters with $s = 5$ and $t = 7$ are mentioned in [2]. Anyway, it would be interesting to try to classify antipodal spherical t -designs with $t \geq 2s - 3$, in particular to show the nonexistence for large t (and hence for large s .)

Appendix 1.

Eigenmatrices and the intersection matrices of the Q-polynomial association schemes attached to the spherical designs X on S^{n-1} of degree 4 and strength 5. We note that $|X| = 2N$, $n \geq 3$, $\frac{n(n+1)}{2} < N \leq \frac{n(n+1)(n+2)}{6}$.

$$B_0 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$B_2 =$$

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ \frac{(N-n)^2(n+2)}{n(3N-n^2-2n)} & 0 & \frac{(N-n)(N^2n+8N^2-9n^2N-18nN+2n^4+8n^3+8n^2)}{2n(3N-n^2-2n)^2} - \frac{N-2n}{2n} \sqrt{\frac{(n+2)(N-n)}{3N-n^2-2n}} \\ 0 & \frac{(N-n)^2(n+2)}{n(3N-n^2-2n)} & \frac{N-2n}{2n} \sqrt{\frac{(n+2)(N-n)}{3N-n^2-2n}} + \frac{(N-n)(N^2n+8N^2-9n^2N-18nN+2n^4+8n^3+8n^2)}{2n(3N-n^2-2n)^2} \\ 0 & 0 & \frac{N^2(n-1)(2N-n^2-n)}{n(3N-n^2-2n)^2} \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{N-2n}{2n} \sqrt{\frac{(n+2)(N-n)}{3N-n^2-2n}} + \frac{(N-n)(N^2n+8N^2-9n^2N-18nN+2n^4+8n^3+8n^2)}{2n(3N-n^2-2n)^2} & \frac{N(-N+n)^2(n+2)}{2n(-3N+n^2+2n)^2} \\ \frac{(N-n)(N^2n+8N^2-9n^2N-18nN+2n^4+8n^3+8n^2)}{2n(3N-n^2-2n)^2} - \frac{N-2n}{2n} \sqrt{\frac{(n+2)(N-n)}{3N-n^2-2n}} & \frac{N(N-n)^2(n+2)}{2n(3N-n^2-2n)^2} \\ \frac{N^2(n-1)(2N-n^2-n)}{n(3N-n^2-2n)^2} & \frac{(N-n)^2(n+2)(2N-n^2-2n)}{n(3N-n^2-2n)^2} \end{bmatrix}$$

$$B_3 =$$

$$\begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 1 \\
0 & \frac{(N-n)^2(n+2)}{n(3N-n^2-2n)} & \frac{N-2n}{2n} \sqrt{\frac{(n+2)(N-n)}{3N-n^2-2n}} + \frac{(N-n)(N^2n+8N^2-9n^2N-18nN+2n^4+8n^3+8n^2)}{2n(3N-n^2-2n)^2} \\
\frac{(N-n)^2(n+2)}{n(3N-n^2-2n)} & 0 & \frac{(N-n)(N^2n+8N^2-9n^2N-18nN+2n^4+8n^3+8n^2)}{2n(3N-n^2-2n)^2} - \frac{N-2n}{2n} \sqrt{\frac{(n+2)(N-n)}{3N-n^2-2n}} \\
0 & 0 & \frac{N^2(n-1)(2N-n^2-n)}{n(3N-n^2-2n)^2} \\
1 & 0 & 0 \\
0 & 0 & 0 \\
\frac{(N-n)(N^2n+8N^2-9n^2N-18nN+2n^4+8n^3+8n^2)}{2n(3N-n^2-2n)^2} - \frac{N-2n}{2n} \sqrt{\frac{(n+2)(N-n)}{3N-n^2-2n}} & \frac{N(N-n)^2(n+2)}{2n(3N-n^2-2n)^2} & \frac{N(N-n)^2(n+2)}{2n(3N-n^2-2n)^2} \\
\frac{N-2n}{2n} \sqrt{\frac{(n+2)(N-n)}{3N-n^2-2n}} + \frac{(N-n)(N^2n+8N^2-9n^2N-18nN+2n^4+8n^3+8n^2)}{2n(3N-n^2-2n)^2} & \frac{N(N-n)^2(n+2)}{2n(3N-n^2-2n)^2} & \frac{(N-n)^2(n+2)(2N-n^2-2n)}{n(3N-n^2-2n)^2} \\
\frac{N^2(n-1)(2N-n^2-n)}{n(3N-n^2-2n)^2} & \frac{(N-n)^2(n+2)(2N-n^2-2n)}{n(3N-n^2-2n)^2} &
\end{bmatrix}$$

$$B_4 = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & \frac{N^2(n-1)(2N-n^2-n)}{n(3N-n^2-2n)^2} \\
0 & 0 & -\frac{N^2(n-1)(-2N+n^2+n)}{n(-3N+n^2+2n)^2} \\
\frac{2N(n-1)(2N-n^2-n)}{n(3N-n^2-2n)} & \frac{2N(n-1)(-2N+n^2+n)}{n(-3N+n^2+2n)} & \frac{2N(n-1)(2N-n^2-n)(2N-n^2-2n)}{n(3N-n^2-2n)^2} \\
0 & 1 & 1 \\
0 & 1 & 1 \\
\frac{N^2(n-1)(2N-n^2-n)}{n(3N-n^2-2n)^2} & \frac{(n+2)(N-n)^2(2N-n^2-2n)}{n(3N-n^2-2n)^2} & \frac{(n+2)(N-n)^2(2N-n^2-2n)}{n(3N-n^2-2n)^2} \\
\frac{N^2(n-1)(2N-n^2-n)}{n(3N-n^2-2n)^2} & \frac{(n+2)(N-n)^2(2N-n^2-2n)}{n(3N-n^2-2n)^2} & \frac{2(4N^2n-4n^3N+6n^2N+n^5-5n^3-10N^2+10nN-2n^2)N}{n(3N-n^2-2n)^2} \\
\frac{2N(n-1)(2N-n^2-n)(2N-n^2-2n)}{n(3N-n^2-2n)^2} & \frac{2(4N^2n-4n^3N+6n^2N+n^5-5n^3-10N^2+10nN-2n^2)N}{n(3N-n^2-2n)^2} &
\end{bmatrix}$$

$$P = \begin{bmatrix} 1 & 1 & \frac{(N-n)^2(n+2)}{n(3N-n^2-2n)} & \frac{(N-n)^2(n+2)}{n(3N-n^2-2n)} & \frac{2(n-1)N(2N-n^2-n)}{n(3N-n^2-2n)} \\ 1 & -1 & -\frac{N-n}{n}\sqrt{\frac{(n+2)(N-n)}{3N-n^2-2n}} & \frac{N-n}{n}\sqrt{\frac{(n+2)(N-n)}{3N-n^2-2n}} & 0 \\ 1 & 1 & \frac{(N-n)(2N-n^2-2n)}{n(3N-n^2-2n)} & \frac{(N-n)(2N-n^2-2n)}{n(3N-n^2-2n)} & -\frac{2N(2N-n^2-n)}{n(3N-n^2-2n)} \\ 1 & -1 & \sqrt{\frac{(n+2)(N-n)}{3N-n^2-2n}} & -\sqrt{\frac{(n+2)(N-n)}{3N-n^2-2n}} & 0 \\ 1 & 1 & -\frac{(n+2)(N-n)}{3N-n^2-2n} & -\frac{(n+2)(N-n)}{3N-n^2-2n} & \frac{2(n-1)N}{3N-n^2-2n} \end{bmatrix}$$

$$Q = \begin{bmatrix} 1 & n & \frac{(n+2)(n-1)}{2} & N-n & \frac{2N-n^2-n}{2} \\ 1 & -n & \frac{(n+2)(n-1)}{2} & -(N-n) & \frac{2N-n^2-n}{2} \\ 1 & -n\sqrt{\frac{3N-n^2-2n}{(n+2)(N-n)}} & \frac{(n-1)(2N-n^2-2n)}{2(N-n)} & n\sqrt{\frac{3N-n^2-2n}{(n+2)(N-n)}} & -\frac{(2N-n^2-n)n}{2(N-n)} \\ 1 & n\sqrt{\frac{3N-n^2-2n}{(n+2)(N-n)}} & \frac{(n-1)(2N-n^2-2n)}{2(N-n)} & -n\sqrt{\frac{3N-n^2-2n}{(n+2)(N-n)}} & -\frac{(2N-n^2-n)n}{2(N-n)} \\ 1 & 0 & -\frac{n+2}{2} & 0 & \frac{n}{2} \end{bmatrix}$$

$$B_1^* = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ n & 0 & \frac{2n}{n+2} & 0 & 0 \\ 0 & n-1 & 0 & \frac{(n-1)n^2}{2(N-n)} & 0 \\ 0 & 0 & \frac{n^2}{n+2} & 0 & n \\ 0 & 0 & 0 & \frac{(2N-n^2-n)n}{2(N-n)} & 0 \end{bmatrix}$$

Appendix 2.

Association schemes attached to the spherical design X in S^{n-1} of degree 4 and strength 5 with $n = 2^{2r}$, $r \geq 1$ and $|X| = 2N = 2^{4r} + 2^{2r+1}$

$$B_0 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$B_2 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 2^{4r-1} & 0 & 2^{r-2}(2^{2r}-2)(2^r-1) & 2^{r-2}(2^{2r}-2)(2^r+1) & 2^{4r-1} \\ 0 & 2^{4r-1} & 2^{r-2}(2^{2r}-2)(2^r+1) & 2^{r-2}(2^{2r}-2)(2^r-1) & 2^{4r-1} \\ 0 & 0 & 2^{2r}-1 & 2^{2r}-1 & 0 \end{bmatrix}$$

$$\begin{aligned}
B_3 &= \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 2^{4r-1} & 2^{r-2}(2^{2r}-2)(2^r+1) & 2^{r-2}(2^{2r}-2)(2^r-1) & 2^{4r-1} \\ 2^{4r-1} & 0 & 2^{r-2}(2^{2r}-2)(2^r-1) & 2^{r-2}(2^{2r}-2)(2^r+1) & 2^{4r-1} \\ 0 & 0 & 2^{2r}-1 & 2^{2r}-1 & 0 \end{bmatrix}, \\
B_4 &= \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 2^{2r}-1 & 2^{2r}-1 & 0 \\ 0 & 0 & 2^{2r}-1 & 2^{2r}-1 & 0 \\ 2(2^{2r}-1) & 2(2^{2r}-1) & 0 & 0 & 2(2^{2r}-2) \end{bmatrix}, \\
P &= \begin{bmatrix} 1 & 1 & 2^{4r-1} & 2^{4r-1} & 2(2^{2r}-1) \\ 1 & -1 & -2^{3r-1} & 2^{3r-1} & 0 \\ 1 & 1 & 0 & 0 & -2 \\ 1 & -1 & 2^r & -2^r & 0 \\ 1 & 1 & -2^{2r} & -2^{2r} & 2(2^{2r}-1) \end{bmatrix}, \\
Q &= \begin{bmatrix} 1 & 2^{2r} & (2^{2r}-1)(2^{2r-1}+1) & 2^{4r-1} & 2^{2r-1} \\ 1 & -2^{2r} & (2^{2r}-1)(2^{2r-1}+1) & -2^{4r-1} & 2^{2r-1} \\ 1 & -2^r & 0 & 2^r & -1 \\ 1 & 2^r & 0 & -2^r & -1 \\ 1 & 0 & -2^{2r-1}-1 & 0 & 2^{2r-1} \end{bmatrix}, \\
B_1^* &= \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 2^{2r} & 0 & \frac{2^{2r}}{2^{2r-1}+1} & 0 & 0 \\ 0 & 2^{2r}-1 & 0 & 2^{2r}-1 & 0 \\ 0 & 0 & \frac{2^{4r}}{2^{2r}+2} & 0 & 2^{2r} \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}
\end{aligned}$$

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